

Higher-Dimensional Complex Dynamics and Spectral Zeta Functions of Fractal Differential Sturm–Liouville Operators

Nishu Lal, Michel L. Lapidus

Department of Mathematics, University of California, Riverside, CA 92521-0135, USA

email: nishul@math.ucr.edu, lapidus@math.ucr.edu

February 21, 2012

Abstract

We investigate the spectral zeta function of a self-similar Sturm–Liouville operator associated with a fractal self-similar measure on the half-line and C. Sabot’s work connecting the spectrum of this operator with the iteration of a rational map of several complex variables. We obtain a factorization of the spectral zeta function expressed in terms of the zeta function associated with the dynamics of the corresponding renormalization map, viewed as a rational function on the complex projective plane, $\mathbb{P}^2(\mathbb{C})$. The result generalizes to several complex variables and to the case of fractal Sturm–Liouville operators a factorization formula obtained by the second author for the spectral zeta function of a fractal string and later extended to the Sierpinski gasket and some other decimable fractals by A. Teplyaev. As a corollary, in the very special case when the underlying self-similar measure is Lebesgue measure on $[0, 1]$, we obtain a representation of the Riemann zeta function in terms of the dynamics of a certain polynomial in $\mathbb{P}^2(\mathbb{C})$, thereby extending to several variables an analogous result by A. Teplyaev. The above fractal Hamiltonians and their spectra are relevant to the study of diffusions on fractals and to aspects of condensed matters physics, including to the key notion of density of states.

2010 Mathematics Subject Classification. Primary 28A80, 31C25, 32A20, 34B09, 34B40, 34B45, 37F10, 37F25, 58J15, 82D30. Secondary 30D05, 32A10, 94C99.

Key words and phrases. Analysis on fractals, fractal Sturm–Liouville operators, self-similar measures and Dirichlet forms, decimation method, renormalization operator and its iterates, multivariable complex dynamics, spectral zeta function, Dirac hyperfunction, Riemann zeta function.

PACS numbers. 02.30.Cj, 02.30.Em, 02.30.Fm, 02.50.Ga, 02.60.Lj, 02.70.Hm, 05.10.Cc, 05.45.Df, 05.60.Gg, 73.20.At.

The research of M. L. Lapidus was partially supported by the National Science Foundation under the research grants DMS-0707524 and DMS-1107750.

1 Introduction

Connections between analysis on fractals, spectral theory, and complex dynamics have been of great interest in physics and mathematics in the past few decades. See, for example, [1]–[3], [5], [13], [26]–[29] and [4], [6], [9]–[11], [14]–[20], [24], [30]–[33], [36], [37], [39], [40], as well as the references therein. In this paper, we further investigate these connections by focusing on the model studied by C. Sabot ([30]–[32]) in the context of multivariable complex dynamics. It involves singular diffusions as well as fractal Hamiltonians and their spectra, along with the associated complex dynamics.

More specifically, we study the Sturm–Liouville operator associated with a fractal self-similar measure on the unit interval and its spectral properties associated with the renormalization map on the complex projective plane, $\mathbb{P}^2(\mathbb{C})$, induced by the decimation method. The decimation method is a process that describes the interesting relations between the spectrum of a differential operator on a suitable self-similar fractal and the dynamics of the iteration of some complex polynomial (or rational function). We show that the spectral zeta function of the operator has a special factorization involving the Dirac hyperfunction and a zeta function associated with the renormalization map. We define this latter zeta function and observe that in a very special case corresponding to Lebesgue measure on $[0, 1]$, the Riemann zeta function can be represented in terms of it.

The theory of fractal strings and the corresponding geometric and spectral zeta functions were first studied by the second author and his collaborators in [18]–[23]; see also [24] for a detailed exposition. A fractal string, \mathcal{L} , is a countable collection of disjoint intervals of lengths ℓ_j . The spectral zeta function of the Dirichlet Laplacian L on \mathcal{L} has the following factorization

$$\zeta_L(s) = \pi^{-s} \zeta(s) \zeta_{\mathcal{L}}(s), \quad (1.1)$$

where $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s$ is the geometric zeta function of the fractal string \mathcal{L} and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function (or its meromorphic continuation). See [19], [20], [22], [23] and [24], Theorem 1.19.

Later on, A. Teplyaev ([39], [40]) studied the spectral zeta function of the Laplacian on a class of symmetric finitely ramified fractals which includes the classic Sierpinski gasket. He discovered that the factorization of the spectral zeta function involves a new zeta function associated with a polynomial (see [40]), thereby naturally extending the factorization for fractal strings. For $\operatorname{Re}(s) > d_R = \frac{2 \log N}{\log c}$, he defined the zeta function of a complex polynomial of degree N , $R = R(z)$, to be

$$\zeta_{R, z_0}(s) = \lim_{n \rightarrow \infty} \sum_{z \in R^{-n} \{z_0\}} (c^n z)^{-\frac{s}{2}},$$

assuming that \mathcal{J} , the Julia set of R , satisfies $\mathcal{J} \subset [0, \infty)$, $R(0) = 0$ and $c = R'(0) > 1$. He then showed that this zeta function has a meromorphic continuation to the half-plane. In addition, he proved that in a special case corresponding to the unit interval, the Riemann zeta function can be written in terms of this zeta function. (See [40] and §2 below.)

Providing a generalization of the decimation method for a certain class of fractals, Sabot ([30]–[33]) introduced a rational function of several complex variables, ρ , called the

renormalization map. In particular, following [30]–[32], we consider the Sturm–Liouville operator $H = -\frac{d}{dm}\frac{d}{dx}$ on the interval $I = [0, 1]$, where m is induced by a self-similar measure. The Sturm–Liouville operator $H_{<\infty>}$ on $[0, \infty)$ is viewed as a limit of the sequence of operators $H_{<n>} = -\frac{d}{dm_{<n>}}\frac{d}{dx}$ with Dirichlet boundary condition on $I_{<n>} = [0, \alpha^{-n}]$ which are the infinitesimal generators of the Dirichlet forms $(a_{<n>}, m_{<n>})$. (Here, $\alpha \in (0, 1)$ is a suitable parameter; see §3.1. Furthermore, note that for $n = 0$, $H_{<0>} = H$ and $I_{<0>} = I$.) The invariant curve ϕ of the map ρ is defined in terms of the trace of the Dirichlet form on a finite set. The dynamics of ρ on this invariant curve is used to compute the spectrum of the corresponding operator. (See §3.1 for a review of the general framework and for a discussion of the associated decimation method.)

In the present paper (see §3.3), we first define (for $\text{Re}(s)$ sufficiently large) the zeta function ζ_ρ associated with the renormalization map on the complex projective plane $\mathbb{P}^2(\mathbb{C})$ as

$$\zeta_\rho(s) = \sum_{p=0}^{\infty} \sum_{\{\lambda \in \mathbb{C}: \rho^p(\phi(\gamma^{-(p+1)}\lambda)) \in D\}} (\gamma^p \lambda)^{-\frac{s}{2}},$$

where D is a suitable subset of the Fatou set of ρ , and relate it to the spectral zeta function ζ_{sp} of the Sturm–Liouville operator, via a product formula of the form

$$\zeta_{sp}(s) = \zeta_\rho(s) \zeta_{\mathcal{L}}(s). \quad (1.2)$$

Here, $\zeta_{\mathcal{L}}$ is the geometric zeta function of some underlying fractal string $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, viewed as a sequence of scales naturally associated with the Sturm–Liouville problem.

Each element of the sequence of spectral zeta functions $\zeta_{H_{<n>}}$ of $H_{<n>}$ on $I_{<n>} = [0, \alpha^{-n}]$ has a product formula, and the most interesting cases are $\zeta_{H_{<0>}}$ and $\zeta_{H_{<\infty>}}$ on I and $\mathbb{R}^+ = [0, \infty)$, respectively. Indeed, we will show that the zeta function $\zeta_{H_{<0>}}(s)$ is exactly equal to $\zeta_\rho(s)$,

$$\zeta_{H_{<0>}}(s) := \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} (\gamma^p \lambda_n)^{-\frac{s}{2}} = \zeta_\rho(s).$$

We will also show that due to the spectral behavior of $H_{<\infty>}$ on \mathbb{R}^+ , the product formula of the spectral zeta function $\zeta_{H_{<\infty>}}$ is the product of a suitable hyperfunction (see §3.2) and of the zeta function produced by the generating set of the spectrum associated with ρ ; see Theorem 3.9.

The ordinary Dirichlet Laplacian $-\frac{d^2}{dx^2}$ is a special case of the Sturm–Liouville operator when $\alpha = \frac{1}{2}$ and hence, m is the Lebesgue measure. Furthermore, as we will see, in this special case, the Riemann zeta function ζ can be expressed in terms of the zeta function, ζ_ρ , associated with the renormalization map ρ . This result extends to several complex variables the corresponding result by A. Teplyaev stating that the Riemann zeta function can be written in terms of the zeta function of a polynomial of one complex variable.

We expect that some of the results obtained in this paper and the techniques developed in the process should be relevant to the study of diffusion on fractals [4] and aspects of condensed matters physics (see, e.g., [1], [5], [13], [26]–[29]), including the key notion of density of states.

2 The spectral zeta function and the zeta function of a polynomial of one variable

We devote this section to the Sierpinski gasket SG , a classical example of a self-similar fractal on which the Laplacian is widely explored. Kigami ([14], [15]) has defined the Laplacian on SG (or on more general p.c.f. self-similar fractals) as a limit of Laplacians on a sequence of approximating finite graphs Γ_m . Given a function on Γ_m (resp., SG), define the renormalized graph energy

$$\mathcal{E}_m(u, u) = \left(\frac{5}{3}\right)^m \sum_{x \sim y} (u(x) - u(y))^2$$

and $\mathcal{E}(u, u) = \lim_{m \rightarrow \infty} \mathcal{E}_m(u, u)$. (Here, the sum is extended to all unordered pairs $\{x, y\}$ of neighboring vertices on Γ_m .) Given the natural self-similar measure μ on SG , the equation $\Delta_\mu u = f$ can be interpreted variationally as $\mathcal{E}(u, v) = - \int_{SG} f v d\mu$, for all suitable functions v vanishing on the boundary points. The pointwise formula for the Laplacian Δ_μ is given by

$$\Delta_\mu u(x) = \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \Delta_m u(x).$$

The physicists R. Rammal [28] and R. Rammal and G. Toulouse [29] have studied the spectrum of the Laplacian, in particular, the eigenvalue equation $\Delta_\mu u = \lambda u$, and discovered the decimation method which establishes the relations between the spectrum of the Laplacian and the dynamics of the iteration of some polynomial R on \mathbb{C} . Later on, T. Shima [36] and M. Fukushima and T. Shima [11] gave a precise mathematical statement of their results, as follows (see also the exposition in [37]):

Theorem 2.1 (Fukushima–Shima, [11], [36]).

- (i) If u is an eigenfunction of $-\Delta_{m+1}$ with eigenvalue λ , that is, $-\Delta_{m+1}u = \lambda u$, and if $\lambda \notin B$, then $-\Delta_m(u|_{V_m}) = R(\lambda)u|_{V_m}$, where $B = \{\frac{5}{4}, \frac{1}{2}, \frac{3}{2}\}$ is the set of ‘forbidden’ eigenvalues.
- (ii) If $-\Delta_m u = R(\lambda)u$ and $\lambda \notin B$, then there exists a unique extension \tilde{u} of u such that $-\Delta_{m+1}\tilde{u} = \lambda\tilde{u}$.

In [40], Teplyaev studied the spectral properties of the Laplacian on SG and explored interesting connections between the spectral zeta function and the iteration of the polynomial R (of a single complex variable) induced by the decimation method. (See also the expository article [39].) The *spectral zeta function* is the meromorphic continuation of the Dirichlet series defined in terms of the eigenvalues of the Laplace operator, as we now explain.

Definition 2.2. The spectral zeta function of a positive self-adjoint operator L with compact resolvent (and hence, with discrete spectrum) is given (for $\text{Re}(s)$ sufficiently large) by

$$\zeta_L(s) = \sum_{j=1}^{\infty} (\kappa_j)^{-s/2}, \tag{2.1}$$

where the positive real numbers κ_j are the eigenvalues of the operator L written in nondecreasing order and counted according to their multiplicities.

The next definition is introduced in [40].

Definition 2.3. Let R be a polynomial of degree N satisfying $R(0) = 0$, $c := R'(0) > 1$, and with Julia set $\mathcal{J} \subset [0, \infty)$. Then the zeta function of R is defined by

$$\zeta_{R,z_0}(s) = \lim_{n \rightarrow \infty} \sum_{z \in R^{-n}\{z_0\}} (c^n z)^{-\frac{s}{2}},$$

for $\operatorname{Re}(s) > d_R := \frac{2 \log N}{\log c}$. Here, R^{-n} denotes the n th inverse iterate of R .

In addition, in the case of the Laplacian on the compact Sierpinski gasket, he discovered the product structure of the spectral zeta function that involves the zeta function of a polynomial. Moreover, revisiting the example of fractal strings, he showed that the Riemann zeta function could be represented as the zeta function of a certain quadratic polynomial, thereby reinterpreting the corresponding product formula (1.1) for the spectral zeta function of a fractal string obtained in [19], [20]; see Remark 2.5 below.

Theorem 2.4 (Teplyaev, [40]). *The spectral zeta function of the Laplacian on SG is*

$$\zeta_{\Delta_\mu}(s) = \zeta_{R,\frac{3}{4}}(s) \frac{5^{-\frac{s}{2}}}{2} \left(\frac{1}{1 - 3 \cdot 5^{-\frac{s}{2}}} + \frac{3}{1 - 5^{-\frac{s}{2}}} \right) + \zeta_{R,\frac{5}{4}}(s) \frac{5^{-s}}{2} \left(\frac{3}{1 - 3 \cdot 5^{-\frac{s}{2}}} - \frac{1}{1 - 5^{-\frac{s}{2}}} \right),$$

where $R(z) = z(5 - 4z)$. Furthermore, there exists $\epsilon > 0$ such that $\zeta_{\Delta_\mu}(s)$ has a meromorphic continuation for $\operatorname{Re}(s) > -\epsilon$, with poles contained in $\left\{ \frac{2in\pi}{\log 5}, \frac{\log 9 + 2in\pi}{\log 5} : n \in \mathbb{Z} \right\}$.

We point out that some time later, and motivated in part by the results and conjectures of [14], [19], [20], but in part independently of [39], [40], Derfel, Grabner, and Vogel [6] have also worked on the same zeta function associated with a polynomial and proved that it has a meromorphic continuation on the whole complex plane. They expressed the spectral zeta function in terms of this zeta function and a zeta function related to the generating set associated with the multiplicities of the eigenvalues of the operator.

In closing this section, we mention the interesting example of the Cantor self-similar fractal string, which is the complement of the middle third Cantor set in $[0, 1]$. Hence, its geometric zeta function is given by

$$\zeta_{\mathcal{L}}(s) = \frac{1}{3^s - 2}$$

and so the factorization formula for the associated spectral zeta function is

$$\zeta_L(s) = \pi^{-s} \zeta(s) \frac{1}{3^s - 2}.$$

This product formula is similar to the one obtained for the spectral zeta function on SG . More generally, one has analogous, although more complicated, expressions for $\zeta_{\mathcal{L}}$, the geometric zeta function of an arbitrary self-similar fractal string (see [24], Chapters 2 and 3), and hence, analogous factorization formulas for ζ_L discussed at the beginning of this introduction and obtained in [19], [20], [24]; see Equation (1.1).

Remark 2.5. By considering the case of the unit interval, A. Teplyaev [40] proved that the Riemann zeta function can be described in terms of the zeta function of a quadratic polynomial of one complex variable. More specifically, the Riemann zeta function $\zeta(s)$ can be represented by

$$\zeta(s) = \frac{1}{2} C^s \zeta_{R,0}(s),$$

where $C = \sqrt{2}\pi$ and $\zeta_{R,0}(s)$ is the zeta function of the polynomial $R(z) = 2z(2 - z)$.

3 The Sturm–Liouville Operator

3.1 Dirichlet forms and the Sturm–Liouville operator on $[0,1]$

We investigate a class of self-similar sets and measures in terms of the spectrum and the spectral zeta function of the associated fractal differential operators. C. Sabot, in a series of papers ([30]–[32]), extended the decimation method to Laplacians defined on a class of finitely-ramified self-similar sets with blow-ups. We discuss the prototypical example he studied, fractal Laplacians on the blow-up $I_{<\infty>} = [0, \infty)$ of the unit interval $I = I_{<0>} = [0, 1]$. From now on, we will assume that

$$0 < \alpha < 1, \quad b = 1 - \alpha, \quad \delta = \frac{\alpha}{1 - \alpha}, \quad \text{and} \quad \gamma = \frac{1}{\alpha(1 - \alpha)}. \quad (3.1)$$

Consider the contraction mappings from $I = [0, 1]$ to itself given by

$$\Psi_1(x) = \alpha x, \quad \Psi_2(x) = 1 - (1 - \alpha)(1 - x),$$

and the unique self-similar measure m on $[0, 1]$ such that for all $f \in C([0, 1])$,

$$\int_0^1 f dm = b \int_0^1 f \circ \Psi_1 dm + (1 - b) \int_0^1 f \circ \Psi_2 dm. \quad (3.2)$$

Define $H_{<0>} = -\frac{d}{dm} \frac{d}{dx}$, the free Hamiltonian with Dirichlet boundary conditions on $[0, 1]$, by $H_{<0>} f = g$ on the domain

$$\left\{ f \in L^2(I, m), \exists g \in L^2(I, m), f(x) = cx + d + \int_0^x \int_0^y g(z) dm(z) dy, f(0) = f(1) = 0 \right\}.$$

The operator $H_{<0>}$ is the infinitesimal generator associated with the Dirichlet form (a, \mathcal{D}) given by

$$a(f, g) = \int_0^1 f' g' dx, \quad \text{for } f, g \in \mathcal{D},$$

where

$$\mathcal{D} = \{f \in L^2(I, m) : f' \in L^2(I, dx)\}.$$

We see that the Dirichlet form a satisfies the similarity equation

$$a(f) = \alpha^{-1} a(f \circ \Psi_1) + (1 - \alpha)^{-1} a(f \circ \Psi_2), \quad (3.3)$$

where we denote the quadratic form $a(f, f)$ by $a(f)$. (See, e.g., [9] for an exposition.)

Next, extend I to $I_{<n>} = \Psi_1^{-n}(I) = [0, \alpha^{-n}]$, which can be expressed as a self-similar set as follows: $I_{<n>} = \bigcup_{i_1, \dots, i_n} \Psi_{i_1 \dots i_n}(I_{<n>})$, where $(i_1, \dots, i_n) \in \{1, 2\}^n$. Here, we have set $\Psi_{i_1 \dots i_n} = \Psi_{i_n} \circ \dots \circ \Psi_{i_1}$. We define the self-similar measure $m_{<n>}$ as

$$\int_{I_{<n>}} f dm_{<n>} = (1 - \alpha)^{-n} \int_I f \circ \Psi_1^{-n} dm,$$

for all $f \in C(I_{<n>})$. Similarly, the corresponding differential operator, $H_{<n>} = -\frac{d}{dm_{<n>}} \frac{d}{dx}$ on $I_{<n>} = [0, \alpha^{-n}]$, can be defined as the infinitesimal generator of the Dirichlet form $(a_{<n>}, \mathcal{D}_{<n>})$ given by

$$a_{<n>}(f, f) = \int_0^{\alpha^{-n}} (f')^2 dx = \alpha^n a(f \circ \Psi_1^{-n}), \text{ for } f \in \mathcal{D}_{<n>},$$

where

$$\mathcal{D}_{<n>} = \{f \in L^2(I_{<n>}, m_{<n>}) : f' \text{ exists and } f' \in L^2(I_{<n>}, dx)\}.$$

We define $H_{<\infty>}$ as the operator $-\frac{d}{dm_{<\infty>}} \frac{d}{dx}$ with Dirichlet boundary conditions on $I_{<\infty>} = [0, \infty)$. It is clear that the (projective system of) measures $m_{<n>}$ give rise to a measure $m_{<\infty>}$ on $I_{<\infty>}$ since for any $f \in \mathcal{D}_{<n>}$ with $\text{supp}(f) \subset [0, 1]$, $a_{<n>}(f, f) = a(f, f)$ and $\int_{I_{<n>}} f dm_{<n>} = \int_I f dm$. Furthermore, we define the corresponding Dirichlet form $(a_{<\infty>}, \mathcal{D}_{<\infty>})$ by

$$a_{<\infty>}(f, f) = \lim_{n \rightarrow \infty} a_{<n>}(f|_{I_{<n>}}, f|_{I_{<n>}}), \text{ for } f \in \mathcal{D}_{<\infty>},$$

where

$$\mathcal{D}_{<\infty>} = \{f \in L^2(I_{<\infty>}, m_{<\infty>}) : \sup_n a_{<n>}(f|_{I_{<n>}}, f|_{I_{<n>}}) < \infty\}.$$

Clearly, $a_{<\infty>}$ satisfies a self-similar identity analogous to Equation (3.3) and its infinitesimal generator is $H_{<\infty>}$.

The study of the eigenvalue problem

$$H_{<n>} f = -\frac{d}{dm_{<n>}} \frac{d}{dx} f = \lambda f \tag{3.4}$$

for the Sturm–Liouville operator with Dirichlet boundary condition on $I_{<n>}$ revolves around a map ρ , called the *renormalization map*, which is initially defined on a space of quadratic forms associated with the fractal and then, via analytic continuation, on \mathbb{C}^3 . The propagator of the above differential equation is very useful in producing this rational map,

$$\rho([x, y, z]) = [x(x + \delta^{-1}y) - \delta^{-1}z^2, \delta y(x + \delta^{-1}y) - \delta z^2, z^2], \tag{3.5}$$

defined on the complex projective plane $\mathbb{P}^2(\mathbb{C})$. Here, $[x, y, z]$ denote the homogeneous coordinates of a point in $\mathbb{P}^2(\mathbb{C})$, where $(x, y, z) \in \mathbb{C}^3$ is identified with $(\beta x, \beta y, \beta z)$ for any $\beta \in \mathbb{C}$, $\beta \neq 0$. Note that in the present case, ρ is a homogeneous polynomial of total degree two. As we shall see later on, the spectrum of the fractal Sturm–Liouville operator

is intimately related to the iteration of ρ . In the sequel, we shall assume that $\delta < 1$ in order for the spectrum of $H_{<0>}$, $H_{<n>}$ ($n = 1, 2, \dots$) and $H_{<\infty>}$ to be purely discrete.

We define the *propagator* $\Gamma_\lambda(s, t)$ for the eigenvalue problem $-\frac{d}{dm_{<\infty>}}\frac{d}{dx}f = \lambda f$ associated with the operator $H_{<\infty>}$ on $I_{<\infty>} = [0, \infty)$ as a time evolution function which for each $0 \leq s \leq t$ is a 2×2 matrix with nonzero determinant such that the solution of the equation satisfies

$$\begin{bmatrix} f(t) \\ f'(t) \end{bmatrix} = \Gamma_\lambda(s, t) \begin{bmatrix} f(s) \\ f'(s) \end{bmatrix}.$$

Using the self-similarity relations (3.2) and (3.3) satisfied by the measure m and the Dirichlet form a , respectively, and recalling that γ is given by Equation (3.1), we obtain $\Gamma_{<n>,\lambda} = D_{\alpha^n} \circ \Gamma_{\gamma^n \lambda} \circ D_{\alpha^{-n}}$ for the eigenvalue problem $-\frac{d}{dm_{<n>}}\frac{d}{dx}f = \lambda f$, where

$$D_{\alpha^n} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^n \end{bmatrix}.$$

The dynamics of the renormalization map ρ plays a key role in calculating the spectrum of the operators $H_{<n>}$. We introduce the *invariant curve* ϕ which is holomorphic on \mathbb{C} and satisfies the functional equation

$$\rho(\phi(\lambda)) = \phi(\gamma\lambda), \quad (3.6)$$

for all $\lambda \in \mathbb{C}$. An *attractive fixed point* x_0 of ρ is a point such that $\rho x_0 = x_0$ and for any other point x in some neighborhood of x_0 , the sequence $\{\rho^n x\}_{n=0}^\infty$ converges to x_0 . The *basin of attraction* of a fixed point is contained in the Fatou set of ρ . For $\delta > 1$, $x_0 = [0, 1, 0]$ is an attractive fixed point of ρ . The set

$$D = \{[x, y, z] : x + \delta^{-1}y = 0\} \quad (3.7)$$

is part of the Fatou set of ρ since it is contained in the basin of attraction of x_0 . (For various notions of higher-dimensional complex dynamics, see, e.g., [7], [8].) The set D and the invariant curve ϕ of ρ together determine the spectrum of $H_{<n>}$ (and of $H_{<\infty>}$), which is pure point. Moreover, the set of eigenvalues can be described by the set

$$S = \{\lambda \in \mathbb{C} : \phi(\gamma^{-1}\lambda) \in D\}, \quad (3.8)$$

the ‘time intersections’ of the curve $\phi(\gamma^{-1}\lambda)$ with D . It turns out that S is countably infinite and contained in \mathbb{R}^+ . We write $S = \{\lambda_k\}_{k=1}^\infty$, with $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ repeated accordingly to multiplicity. Furthermore, we call S the *generating set* for the spectrum of $H_{<n>}$, with $n = 0, 1, \dots, \infty$.

Let $S_p = \gamma^p S$, for each $p \in \mathbb{Z}$. The spectrum of $H_{<\infty>}$ with Dirichlet boundary conditions is pure point for $\alpha < \frac{1}{2}$ (hence, for $\delta < 1$ and $\gamma \geq 4$), and it can be deduced from the spectrum of $H_{<0>}$, as we now explain.

Theorem 3.1 (Sabot, [32]). *The spectrum of $H_{<0>}$ on $I = I_{<0>}$ is $\bigcup_{p=0}^\infty S_p$ and the spectrum of $H_{<\infty>}$ on \mathbb{R}^+ is $\bigcup_{p=-\infty}^\infty S_p$. Moreover, for any $n \geq 0$, the spectrum of $H_{<n>}$ is equal to $\bigcup_{p=-n}^\infty S_p$.*

The diagram of the set of eigenvalues of the operator $H_{<\infty>}$ is as follows:

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & & \vdots \\
& \gamma^{-2}\lambda_1 & \gamma^{-2}\lambda_2 & \gamma^{-2}\lambda_3 & \gamma^{-2}\lambda_4 & \cdots & & \\
& \gamma^{-1}\lambda_1 & \gamma^{-1}\lambda_2 & \gamma^{-1}\lambda_3 & \gamma^{-1}\lambda_4 & \cdots & & \\
& \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots & & \\
& \gamma\lambda_1 & \gamma\lambda_2 & \gamma\lambda_3 & \gamma\lambda_4 & \cdots & & \\
& \gamma^2\lambda_1 & \gamma^2\lambda_2 & \gamma^2\lambda_3 & \gamma^2\lambda_4 & \cdots & & \\
& \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Sabot's work ([30]–[33]) has sparked an interest in generalizing the decimation method to a broader class of fractals and therefore, to the iteration of rational functions of several complex variables. For each $k \geq 1$, we denote by f_k the solution of the equation $H_{<\infty>}f = \lambda_k f$ for $\lambda_k \in S$. In other words, f_k is an eigenfunction of $H_{<\infty>}$ associated with the eigenvalue $\lambda_k \in S$. (Note that f_k is uniquely determined, up to a nonzero multiplicative constant which can be fixed by a suitable normalization.)

Theorem 3.2 (Sabot, [32]).

- (i) *Given any $k \geq 1$, if f_k is the solution of the equation $H_{<\infty>}f = \lambda_k f$ for $\lambda_k \in S$, then $f_{k,p} := f_k \circ \Psi_1^{-p}$ is the solution of the equation $H_{<\infty>}f = \lambda_{k,p} f$, where $\lambda_{k,p} := \gamma^p \lambda_k$ and $p \in \mathbb{Z}$ is arbitrary.*
- (ii) *Moreover, if $f_{k,p} = f_k \circ \Psi_1^{-p}$ is the solution of the equation $H_{<\infty>}f = \lambda_{k,p} f$, then $f_{k,p,<n>} := f_{k,p}|_{I_{<n>}}$, the restriction of $f_{k,p}$ to $I_{<n>}$, is the solution of the equation $H_{<n>}f = \lambda_{k,p} f$.*

We now describe the relations between the eigenvalues of two consecutive operators $H_{<n>}$ and $H_{<n+1>}$. As above, we denote by $f_{k,p,<n>}$ the restriction of $f_{k,p}$ to $I_{<n>}$.

Lemma 3.3. *For each fixed $n \geq 0$, the set $\{f_{k,p,<n>} : k \geq 1, p \geq -n\}$ is a complete set of eigenfunctions of $H_{<n>}$ in the complex Hilbert space $L^2(\mathbb{R}^+, m_{<\infty>})$.*

There are a number of fractals for which the decimation method has been established or explored. The interested readers can consult the following references by Shima [36], Fukushima–Shima [11], Kigami–Lapidus [17], Strichartz [37], Bajorin *et al.* ([2], [3]), Teplyaev ([39], [40]) and Derfel *et al.* [6] in the case of rational functions of a single complex variable, and by Sabot ([30]–[33]) in the significantly more general case of rational functions of several complex variables.

3.2 Hyperfunctions

In this section, we will give a brief introduction to hyperfunctions, which are the distributional generalization of analytic functions. (We refer the interested reader to M. Sato's papers ([34], [35]) for a general discussion of this beautiful subject, and to the books by U. Graf [12] and M. Morimoto [25] for a more elementary and directly accessible introduction.)

Let Ω be a subset of \mathbb{R} . A complex neighborhood of Ω is an open subset $U \subset \mathbb{C}$ such that Ω is a closed subset of U . We denote by $\mathcal{C}(U)$ and $\mathcal{C}(U \setminus \Omega)$ the vector spaces of holomorphic functions on U and $U \setminus \Omega$, respectively. The quotient space $\mathcal{C}(U \setminus \Omega)/\mathcal{C}(U)$ is equipped with the equivalence relation according to which any holomorphic function in $\mathcal{C}(U \setminus \Omega)$ that extends holomorphically to all of U is identified with the zero function. By definition, each equivalence class represents a *hyperfunction*.

Alternatively, a hyperfunction on the real line, $f(x) = [F(z)] = [F_+, F_-]$, consists of two functions, $F_+(z)$ and $F_-(z)$, which are analytic in the upper and the lower half-planes, respectively, and such that the following limit exists:

$$\lim_{\epsilon \rightarrow 0^+} \left(F_+(x + i\epsilon) - F_-(x - i\epsilon) \right).$$

Every hyperfunction $[F_+, F_-]$ forms an equivalence class of the form $[h + F_+, h + F_-]$, where h is a holomorphic function on U .

Any holomorphic function g can be expressed as a hyperfunction $g = [g, 0] = [0, -g]$. All the standard elementary operations on hyperfunctions are satisfied, such as

$$[F_+, F_-] + [G_+, G_-] = [F_+ + G_+, F_- + G_-]$$

and

$$\frac{d}{dx}[F_+, F_-] = \left[\frac{d}{dx}F_+, \frac{d}{dx}F_- \right].$$

However, the product of two hyperfunctions does not always make sense but the product of a hyperfunction F by a holomorphic function h is well defined and is given by $h[F_+, F_-] = [hF_+, hF_-]$.

Example 3.4. A hyperfunction $f(x)$ is a function on the real line described as a difference of two holomorphic functions defined on the upper and lower half-planes. Consider the following functions:

$$\begin{aligned} I_1(z) &= \begin{cases} 1, & \operatorname{Im}(z) > 0 \\ 0, & \operatorname{Im}(z) < 0, \end{cases} \\ I_2(z) &= \begin{cases} 0, & \operatorname{Im}(z) > 0 \\ -1, & \operatorname{Im}(z) < 0, \end{cases} \\ I_3(z) &= \begin{cases} \frac{1}{2}, & \operatorname{Im}(z) > 0 \\ -\frac{1}{2}, & \operatorname{Im}(z) < 0. \end{cases} \end{aligned}$$

These three functions define the same hyperfunction $f(x) = 1$, which is the ordinary constant function. (See [12].)

Example 3.5. *This example highlights one of the most important hyperfunctions; it is called the Dirac delta hyperfunction on the real line \mathbb{R} and is given by $\delta_{\mathbb{R}}(z) = [-\frac{1}{2\pi iz}, -\frac{1}{2\pi iz}]$. For $x \neq 0$, we have*

$$\begin{aligned} f(x) &= \lim_{\epsilon \rightarrow 0^+} \left(F_+(x + i\epsilon) - F_-(x - i\epsilon) \right) \\ &= \lim_{\epsilon \rightarrow 0^+} \left(\frac{-1}{2\pi i(x + i\epsilon)} - \frac{-1}{2\pi i(x - i\epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{\pi(x^2 + \epsilon^2)} = 0. \end{aligned}$$

For $x = 0$, however, the above limit does not exist, and this is the point at which the delta ‘function’ has an isolated singularity. (See [12], §1.2 and [25], §3.4.)

3.3 The zeta function associated with the renormalization map

We now introduce a multivariable analog of the polynomial zeta function of Definition 2.3.

Definition 3.6. *We define the zeta function of the renormalization map ρ to be*

$$\zeta_{\rho}(s) = \sum_{p=0}^{\infty} \sum_{\{\lambda \in \mathbb{C} : \rho^p(\phi(\gamma^{-(p+1)}\lambda)) \in D\}} (\gamma^p \lambda)^{-\frac{s}{2}}, \quad (3.9)$$

for $\text{Re}(s)$ sufficiently large.

Recall the definition 2.2 of the spectral zeta function $\zeta_L(s)$ of a positive self-adjoint operator L with discrete spectrum.

We can now state our first result:

Theorem 3.7. *The zeta function $\zeta_{\rho}(s)$ of the renormalization map ρ is equal to the spectral zeta function $\zeta_{H_{<0>}}(s) = \sum_{\lambda \in S} \sum_{p=0}^{\infty} (\gamma^p \lambda)^{-\frac{s}{2}}$ of $H_{<0>}(s)$: $\zeta_{\rho}(s) = \zeta_{H_{<0>}}(s)$. (An expression for $\zeta_{H_{<0>}}(s)$ is given by the $n = 0$ case of Proposition 3.8 below.)*

Proof. We have successively:

$$\begin{aligned} \zeta_{\rho}(s) &= \sum_{p=0}^{\infty} \sum_{\{\lambda \in \mathbb{C} : \rho^p(\phi(\gamma^{-(p+1)}\lambda)) \in D\}} (\gamma^p \lambda)^{-\frac{s}{2}} \\ &= \sum_{p=0}^{\infty} \sum_{\lambda \in S} (\gamma^p \lambda)^{-\frac{s}{2}} \\ &= \zeta_{H_{<0>}}(s). \end{aligned}$$

In order to justify the first equality, we show that the set $\{\lambda \in \mathbb{C} : \rho^p(\phi(\gamma^{-(p+1)}\lambda)) \in D\}$ is exactly equal to $S = \{\lambda \in \mathbb{C} : \phi(\gamma^{-1}\lambda) \in D\}$. Recall the relation $\rho(\phi(\lambda)) = \phi(\gamma\lambda)$, for all $\lambda \in \mathbb{C}$. After p iterations, this equation becomes $\rho^p(\phi(\lambda)) = \phi(\gamma^p\lambda)$. Therefore, we get $\rho^p(\phi(\gamma^{-(p+1)}\lambda)) = \phi(\gamma^p\gamma^{-p-1}\lambda) = \phi(\gamma^{-1}\lambda)$, for $p = 0, 1, 2, \dots$ \square

We have a sequence of operators $H_{<n>} = -\frac{d}{dm_{<n>}} \frac{d}{dx}$, starting with $H_{<0>}$ on $[0, 1]$, which converges to the Sturm–Liouville operator $H_{<\infty>}$ on $[0, \infty)$. We will now consider the associated spectral zeta functions and their product formulas. Recall that given an integer $n \geq 0$, the spectral zeta $\zeta_{H_{<n>}}(s)$ of $H_{<n>}$ on $[0, \alpha^{-n}]$ is

$$\zeta_{H_{<n>}}(s) = \sum_{\lambda \in S} \sum_{p=-n}^{\infty} (\gamma^p \lambda)^{-\frac{s}{2}}.$$

Then, a simple computation yields the following result.

Proposition 3.8. *For $n \geq 0$ and $\operatorname{Re}(s)$ sufficiently large and positive, we have*

$$\zeta_{H_{<n>}}(s) = \left[(\gamma^n)^{\frac{s}{2}} + \dots + \gamma^s + \gamma^{\frac{s}{2}} + \sum_{p=0}^{\infty} (\gamma^p)^{-\frac{s}{2}} \right] \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} = \frac{(\gamma^n)^{\frac{s}{2}}}{1 - \gamma^{-\frac{s}{2}}} \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}}. \quad (3.10)$$

Hence, $\zeta_{H_{<n>}}(s) = \frac{(\gamma^n)^{\frac{s}{2}}}{1 - \gamma^{-\frac{s}{2}}} \zeta_S(s)$, where $\zeta_S(s) := \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}}$ (for $\operatorname{Re}(s)$ large enough) or is given by its meromorphic continuation thereof. In the sequel, $\zeta_S(s)$ is called the geometric zeta function of the generating set S .

In the case of the operator $H_{<\infty>}$, the geometric part of the product formula of the spectral zeta function $\zeta_{<\infty>}$, seems to be equal to zero. However, it is an example of a hyperfunction known as the Dirac δ -function, as we shall soon see. For now, we carry out the naive computation as follows:

$$\begin{aligned} \zeta_{H_{<\infty>}}(s) &= \sum_{p=-\infty}^{\infty} (\gamma^p)^{-\frac{s}{2}} \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} \\ &= \left[\sum_{p=-\infty}^{-1} (\gamma^p)^{-\frac{s}{2}} + \sum_{p=0}^{\infty} (\gamma^p)^{-\frac{s}{2}} \right] \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} \\ &= \left[\frac{\gamma^{\frac{s}{2}}}{1 - \gamma^{\frac{s}{2}}} + \frac{1}{1 - \gamma^{-\frac{s}{2}}} \right] \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} \\ &= \left[\frac{1}{1 - \gamma^{-\frac{s}{2}}} - \frac{1}{1 - \gamma^{-\frac{s}{2}}} \right] \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} \\ &= 0. \end{aligned}$$

Note that this computation is meaningless, unless it is properly interpreted. Indeed, we have added two infinite series, one of which is convergent only for $\operatorname{Re}(s) > 0$, whereas the other series is convergent only for $\operatorname{Re}(s) < 0$. In fact, fortunately, the geometric part $\sum_{p=-\infty}^{\infty} (\gamma^p)^{-\frac{s}{2}}$ can be interpreted in terms of the Dirac delta hyperfunction on the unit circle, $\delta_{\mathbb{T}}(w) = [\delta_{\mathbb{T}}^+(w), \delta_{\mathbb{T}}^-(w)]$, by means of a suitable change of variable; namely, $w = \gamma^{-\frac{s}{2}}$. The delta hyperfunction on the unit circle \mathbb{T} is defined as $\delta_{\mathbb{T}} = [\delta_{\mathbb{T}}^+, \delta_{\mathbb{T}}^-] = [\frac{1}{1-z}, \frac{1}{z-1}]$. It consists of two analytic functions, $\delta_{\mathbb{T}}^+ : E \rightarrow \mathbb{C}$ and $\delta_{\mathbb{T}}^- : \mathbb{C} \setminus \bar{E} \rightarrow \mathbb{C}$, where $E = \{z \in \mathbb{C} :$

$|z| < 1 + \frac{1}{N}\}$ for a large natural number N . In other words, a hyperfunction on \mathbb{T} can be viewed as a suitable pair of holomorphic functions, one on the unit disk $|z| < 1$, and one on its exterior, $|z| > 1$. (See, for example, [12], §1.3 and [25], §3.3.2 for a discussion of various changes of variables in a hyperfunction. Moreover, see [38] for a detailed discussion of $\delta_{\mathbb{T}}$ and, more generally, of hyperfunctions on the unit circle \mathbb{T} .)

Theorem 3.9. *The factorization of the spectral zeta function $\zeta_{H_{<\infty>}}(s)$ of $H_{<\infty>}$ is given by*

$$\zeta_{H_{<\infty>}}(s) = \zeta_S(s) \cdot \delta_{\mathbb{T}}(w), \quad (3.11)$$

where $w := \gamma^{-\frac{s}{2}}$ and $\delta_{\mathbb{T}}(w) = [\delta_{\mathbb{T}}^+(w), \delta_{\mathbb{T}}^-(w)]$ is the Dirac delta hyperfunction on the unit circle \mathbb{T} . (Note that $|w| < 1$ for $\operatorname{Re}(s) > 0$ and $|w| > 1$ for $\operatorname{Re}(s) < 0$, since $\gamma \geq 4$ implies that $\log \gamma > 0$.) Furthermore, here, as above, $\zeta_S(s) := \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}}$ is the geometric zeta function of the generating set S . (As before, we continue to denote by $\zeta_S(s)$ the meromorphic continuation of this zeta function, when it exists.)

Proof. Let $w = \gamma^{-\frac{s}{2}}$. Then we have

$$\zeta_{H_{<\infty>}}(s) = \left(\sum_{p=-\infty}^{\infty} (\gamma^p)^{-\frac{s}{2}} \right) \left(\sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}} \right) = \left(\sum_{p=-\infty}^{\infty} w^p \right) \zeta_S(s). \quad (3.12)$$

We introduce the formal expression $\psi(w) = \sum_{p=-\infty}^{\infty} w^p$. Note that

$$\psi(w) = \begin{cases} \sum_{p=0}^{\infty} w^p, & \text{if } |w| < 1 \\ \sum_{p=-\infty}^{-1} w^p, & \text{if } |w| > 1 \end{cases}$$

or equivalently,

$$\psi(w) = \begin{cases} \frac{1}{1-w}, & \text{if } |w| < 1 \\ \frac{1}{w-1}, & \text{if } |w| > 1. \end{cases}$$

Now, we observe that $\psi(w)$ is in fact equal to the Dirac delta hyperfunction $\delta_{\mathbb{T}}(w) = [\delta_{\mathbb{T}}^+(w), \delta_{\mathbb{T}}^-(w)]$ on the unit circle \mathbb{T} . We refer the interested reader to [38] for a precise mathematical discussion of the delta hyperfunction $\delta_{\mathbb{T}}$.

Next, we make the change of variable $w = \gamma^{-\frac{s}{2}}$. (Clearly, $|w| < 1$ and $|w| > 1$ correspond to the upper and lower half-planes $\operatorname{Re}(s) > 0$ and $\operatorname{Re}(s) < 0$, respectively. Indeed, $\log \gamma > 0$ since $\gamma \geq 4$.)

Hence, with the above substitution $w = \gamma^{-\frac{s}{2}}$ and in light of (3.12) and the definition of $\psi(w)$, we have established the relation

$$\zeta_{H_{<\infty>}}(s) = \delta_{\mathbb{T}}(w) \cdot \zeta_S(s),$$

as desired. □

Next, we revisit some of the earlier results obtained in [40]. More precisely, we show that the zeta function associated with the renormalization map coincides with the Riemann zeta function for a special value of α .

3.3.1 The case $\alpha = \frac{1}{2}$: Connection with the Riemann zeta function

When $\alpha = \frac{1}{2}$, the self-similar measure m coincides with Lebesgue measure on $[0, 1]$ and hence, $H = H_{<0>}$ coincides with the usual Dirichlet Laplacian on the unit interval $[0, 1]$.

Theorem 3.10. *When $\alpha = \frac{1}{2}$, the Riemann zeta function ζ is equal (up to a trivial factor) to the zeta function ζ_ρ associated with the renormalization map ρ on $\mathbb{P}^2(\mathbb{C})$. More specifically, we have*

$$\zeta(s) = \pi^s \zeta_\rho(s) = \frac{\pi^s}{1 - 2^{-s}} \zeta(s), \quad (3.13)$$

where ζ_ρ is given by Definition 3.6 and the polynomial map $\rho : \mathbb{P}^2(\mathbb{C}) \rightarrow \mathbb{P}^2(\mathbb{C})$ is given by Equation (3.5) with $\alpha = \frac{1}{2}$ (and hence, in light of (3.1), with $\delta = 1$ and $\gamma = 4$):

$$\rho([x, y, z]) = [x(x + y) - z^2, y(x + y) - z^2, z^2]. \quad (3.14)$$

Proof. First, we consider the case when $n = 0$ in Theorem 3.7 and Proposition 3.8. Then, since $\gamma = 4$ in the present situation, we have

$$\zeta_{H_{<0>}}(s) = \zeta_\rho(s) = \frac{1}{1 - \gamma^{-\frac{s}{2}}} \zeta(s) = \frac{1}{1 - 2^{-s}} \zeta(s). \quad (3.15)$$

Next, we recall that the eigenvalues of the Dirichlet Laplacian $L = -\frac{d^2}{dx^2}$ on $[0, 1]$ are $\kappa_j = \pi^2 j^2$, for $j = 1, 2, \dots$. Therefore, in light of Definition 2.2, the associated spectral zeta function is

$$\zeta_L(s) = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-\frac{s}{2}} = \pi^{-s} \zeta(s),$$

where ζ is the Riemann zeta function. We recall that in the present case, the Sturm–Liouville operator $H_{<0>}$ and the Dirichlet Laplacian L on $[0, 1]$ coincide; hence, the corresponding spectral zeta functions are equal: $\zeta_{H_{<0>}}(s) = \zeta_L(s)$. In light of Theorem 3.7, $\zeta_{H_{<0>}}(s) = \zeta_\rho(s)$ and we therefore obtain the relation

$$\zeta(s) = \pi^s \zeta_\rho(s),$$

with ζ_ρ given by Equation (3.9) and ρ defined by Equation (3.14), as desired. \square

Remark 3.11. *This is an extension to several complex variables of A. Teplyaev’s result [40] discussed in Remark 2.5 above.*

Remark 3.12. *Still assuming that $\alpha = \frac{1}{2}$ and since Equation (3.13) implies that $\zeta_\rho(s) = \pi^{-s} \zeta(s)$, we deduce that the factorization formula (1.1) for the spectral zeta function $\zeta_L(s) = \zeta_{sp}(s)$ of a fractal string \mathcal{L} can be rewritten as follows:*

$$\zeta_L(s) = \zeta_\rho(s) \cdot \zeta_{\mathcal{L}}(s), \quad (3.16)$$

in agreement with formula (1.2). (Compare with [19], [20] and [24], Theorem 1.19.) Here, ρ is the homogeneous quadratic polynomial on \mathbb{C}^3 (or rather, on $\mathbb{P}^2(\mathbb{C})$) given by Equation (3.14).

4 Concluding Remarks

We expect to obtain a similar product structure for other fractal Laplacians on Sabot decimable self-similar fractals. Our immediate aim would be to apply the obtained results to the infinite (or unbounded) Sierpinski gasket, at least when the spectrum is discrete, in which case we expect to recover a similar hyperfunction in the product formula of the spectral zeta function of the Laplacian. In addition, we plan to consider the modified Koch curve for which the decimation method is well established with a rational map of one complex variable. A long-term goal would be to analyze a class of finitely-ramified self-similar sets with blow-ups; a special case of that is the aforementioned infinite Sierpinski gasket, which corresponds to the blow-up of the classic Sierpinski gasket.

Using Sabot's multivariable extension of the decimation method, one should be able to obtain an analogous factorization formula for such fractals. Such a generalization would also enable us to better understand the nature of the spectrum of the Laplacian and to formulate and possibly solve suitable direct and inverse spectral problems in this context. In a more familiar language, and appropriately interpreted, this would enable us in certain situations to “hear the shape of a fractal drum”. (See, e.g., [16]–[24].)

Furthermore, it would be interesting, both mathematically and physically, to obtain related results in the situation where the Laplacian under investigation has a continuous spectrum or, more generally, a combination of continuous and discrete spectra. We would then have to work with a suitably defined notion of density of states, both at the geometric and spectral levels. (Compare, e.g., [24], §6.3.1 and [16], [31].)

Moreover, as we have seen, Sabot discovered in [30]–[33] some striking relationships between the spectral properties of certain differential operators on fractals and the iteration of rational maps of several complex variables. The further study of the connections between these rational maps and the spectral zeta functions of fractal Laplacians is one of the main goals of future research on this topic and should lead to a deeper exploration of complex dynamics in higher dimensions, in relation to the spectral theory of fractal drums. It may also have applications to condensed matter physics ([1], [5], [13], [26]–[29]), particularly, the study of random and fractal media.

Acknowledgements. We wish to thank Michael Maroun for a helpful conversation about hyperfunctions.

References

- [1] Alexander, S., Orbach, R., Density of states on fractals: fractons, *J. Physique Lettres* **43** (1982).
- [2] Bajorin, N., Chen, T., Dagan, A., Emmons, C., Hussein, M., Khalil, M., Mody, P., Steinhurst, B., Teplyaev, A., Vibration modes of 3n-gaskets and other fractals, *J. Phys. A: Math. Theor.* **41** (2008), 015101 (21pp).

- [3] Bajorin, N., Chen, T., Dagan, A., Emmons, C., Hussein, M., Khalil, M., Mody, P., Steinhurst, B., Teplyaev, A., Vibration spectra of finitely ramified, symmetric fractals, *Fractals* **16** (2008), 243–258.
- [4] Barlow, M. T., Random walks and diffusion on fractals, in: *Proc. Intern. Congress Math.* (Kyoto, 1990), vol. II, Springer-Verlag, Berlin and New York, 1991, pp. 1025–1035.
- [5] Berry, M. V., Distribution of modes in fractal resonators, in: *Structural Stability in Physics*, W. Güttinger and H. Eikemeier (eds.), Springer-Verlag, Berlin, 1979, pp. 51–53.
- [6] Derfel, G., Grabner, P., Vogl, F., The zeta function of the Laplacian on certain fractals, *Trans. Amer. Math. Soc.* **360** (2008), 881–897.
- [7] Fornaess, J. E., *Dynamics in Several Complex Variables*, CBMS Conf. Series in Math., vol. 87, Amer. Math. Soc., Providence, RI, 1996.
- [8] Fornaess, J. E., Sibony, N., *Complex dynamics in higher dimension I*, *Astérisque* **222** (1994), 201–231.
- [9] Freiberg, U., A survey on measure geometric Laplacians on Cantor like sets, *Arabian J. Sci. Eng.* **28** (2003), 189–198.
- [10] Fujita, T., A fractional dimension, self-similarity and a generalized diffusion operator, in: *Probabilistic Methods in Mathematical Physics* (Katata and Kyoto, 1985), K. Ito and N. Ikeda (eds.), Proc. Taniguchi Intern. Symp., Toyko: Kinokuniya, 1987, pp. 83–90.
- [11] Fukushima, M., Shima, T., On a spectral analysis for the Sierpinski gasket, *Potential Analysis* **1** (1992), 1–35.
- [12] Graf, U., *Introduction to Hyperfunctions and Their Integral Transforms: An applied and computational approach*, Birkhäuser, Basel, 2010.
- [13] Hattori, K., Hattori, T., Watanabe, H., Gaussian field theories on general networks and the spectral dimensions, *Progr. Theoret. Phys. Suppl.* **92** (1987), 108–143.
- [14] Kigami, J., Harmonic calculus on p.c.f self-similar sets, *Trans. Amer. Math. Soc.* **335** (1993), 721–755.
- [15] Kigami, J., *Analysis on Fractals*, Cambridge Tracts in Mathematics, vol. 143, Cambridge University Press, Cambridge, 2001.
- [16] Kigami, J., Lapidus, M. L., Weyl’s problem for the spectral distribution of Laplacians on p.c.f self-similar fractals, *Commun. Math. Phys.* **158**(1) (1993), 93–125.
- [17] Kigami, J., Lapidus, M. L., Self-similarity of volume measures for Laplacians on p.c.f. self-similar fractals, *Commun. Math. Phys.* **217** (2001), 165–180.

- [18] Lapidus, M. L., Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture, *Trans. Amer. Math. Soc.* **325** (1991), 465–529.
- [19] Lapidus, M. L., Spectral and fractal geometry: From the Weyl–Berry conjecture for the vibrations of fractals drums to the Riemann zeta function, in: *Differential Equations and Mathematical Physics* (Birmingham, AL, 1990), C. Bennewitz (ed.), Math. Sci. Engrg., vol. 186, Academic Press, Boston, MA, 1992, pp. 151–181.
- [20] Lapidus, M. L., *Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl–Berry conjecture*, in: *Ordinary and Partial Differential Equations*, vol. IV, B. D. Sleeman and R. J. Jarvis (eds.), Pitman Research Notes in Math. Series, vol. 289, Longman Scientific and Technical, London, 1993, pp. 126–209.
- [21] Lapidus, M. L., *Fractals and vibrations: Can you hear the shape of a fractal drum?*, *Fractals* **3**, No. 4 (1995), 725–736. (Special issue in honor of Benoit B. Mandelbrot’s 70th birthday.)
- [22] Lapidus, M. L., Maier, H., The Riemann hypothesis and inverse spectral problems for fractal strings, *J. London Math. Soc.* (2) **52** (1995), 15–34.
- [23] Lapidus, M. L., Pomerance, C., The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums, *Proc. London Math. Soc.* (3) **66** (1993), 41–69.
- [24] Lapidus, M. L., van Frankenhuijsen, M., *Fractal Geometry, Complex Dimension and Zeta Functions: Geometry and spectra of fractal strings*, Springer Monographs in Mathematics, Springer, New York, 2006. (Second rev. and enl. ed. to appear in 2012.)
- [25] Morimoto, M., *An Introduction to Sato’s Hyperfunctions*, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, R.I., 1993. (English transl. of Kyoritsu–Shuppan, 1976.)
- [26] Nakayama, T., Yakubo, K., *Fractal Concepts in Condensed Matter Physics*, Springer–Verlag, 2003.
- [27] Olemskoi, A., *Fractals in Condensed Matter Physics*, I. Khalatnikov (ed.), *Phys. Rev.* vol. 18, Part I, Gordon & Breach, London, 1996.
- [28] Rammal, R., Spectrum of harmonic excitations on fractals, *J. de Physique* **45** (1984), 191–206.
- [29] Rammal, R., Toulouse, G., Random walks on fractal structures and percolation cluster, *J. Physique Lettres* **44** (1983), L13–L22.
- [30] Sabot, C., Density of states of diffusions on self-similar sets and holomorphic dynamics in \mathbb{P}^k : the example of the interval $[0, 1]$, *C. R. Acad. Sci. Paris Sér. I: Math.* **327** (1998), 359–364.

- [31] Sabot, C., Integrated density of states of self-similar Sturm–Liouville operators and holomorphic dynamics in higher dimension, *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001), 275–311.
- [32] Sabot, C., Spectral analysis of a self-similar Sturm–Liouville operator, *Indiana Univ. Math. J.* **54** (2005), 645–668.
- [33] Sabot, C., *Spectral properties of self-similar lattices and iteration of rational maps*, Mémoires Soc. Math. France (New Series), No. 92, 2003, 1–104.
- [34] Sato, M., Theory of hyperfunctions, *Sugaku* **10** (1958), 1–27. (Japanese)
- [35] Sato, M., Theory of hyperfunctions I & II, *J. Fac. Sci. Univ. Tokyo*, Sec. IA, **8** (1959), 139–193 & **8** (1960), 387–437.
- [36] Shima, T., On eigenvalue problems for Laplacians on p.c.f. self-similar sets, *Japan J. Indust. Appl. Math.* **13** (1996), 1–23.
- [37] Strichartz, R. S., *Differential Equations on Fractals: A tutorial*, Princeton University Press, Princeton, 2006.
- [38] Taguchi, Y., A characterization of the space of Sato-hyperfunctions on the unit circle, *Hiroshima Math. J.* **17** (1987), 41–46.
- [39] Teplyaev A., *Spectral zeta function of symmetric fractals*, in: *Fractal Geometry and Stochastics III*, Progress in Probability, vol. 57, Birkhäuser-Verlag, Basel, 2004, pp. 245–262.
- [40] Teplyaev, A., Spectral zeta functions of fractals and the complex dynamics of polynomials, *Trans. Amer. Math. Soc.* **359** (2007), 4339–4358.